Division **E**

# Mathematical Olympiads
## November 12, 2019
*For Elementary & Middle Schools*

Contest **1**

*Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.*

*Name:* ______________________________________________

**1A** Add: 8883 + 8838 + 8388 + 3888.

**1B** Five students entered a contest where they guessed how many marbles were in a jar. The person with the closest guess won the contest.

- Ashley guessed 98 marbles.
- Beth guessed 105 marbles.
- Candace guessed 109 marbles.
- Dennis guessed 113 marbles.
- Edward guessed 115 marbles.

Two of the students were 3 away from the actual number of marbles. Who won the contest?

**1C** Each letter in the following addition problem represents a different digit and each of the same letter represents the same digit. No letter can be either 5 or 9.
Determine the sum PLAN in this cryptarithm.

$$\begin{array}{r} 5\ A\ 9 \\ 5\ A\ 9 \\ +\ 5\ A\ 9 \\ \hline P\,L\,A\,N \end{array}$$

*Name:* ________________________________________

| Answer Column |
|---|
| **1A** |
| **1B** |
| **1C** |
| **1D** |
| **1E** |
| *Do Not Write in this Space. For PICO's Use Only.* *SCORE:* |

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**1D** The fraction $\frac{5}{13}$ can be written as the decimal 0.384615384615… where the digits 384615 keep repeating. What is the 2019$^{th}$ digit to the right of the decimal point?

**1E** Khadijah is thinking of a 5-digit number using all the digits 0, 1, 2, 3, and 4. The number is divisible by 5, by 8, and by 11. The leftmost digit of the 5-digit number is not 4. What is the number?

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Division **E**

# Mathematical Olympiads
**December 10, 2019**
*For Elementary & Middle Schools*

Contest **2**

*Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.*

*Name:* ______________________________________________

**2A** Evaluate: 98 – 87 + 76 – 65 + 54 – 43 + 32 – 21.

**2B** In the figure, how many squares (of any size) can be drawn along existing lines such that the number in a square or the sum of the numbers inside each larger square is even?

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

**2C** What is the value of $1974 \times 6 + 1969 \times 4 + 45 \times 6 + 50 \times 4$?

*Name:* ______________________________

| Answer Column |
|---|
| **2A** |
| **2B** |
| **2C** |
| **2D** $ |
| **2E** |
| *Do Not Write in this Space. For PICO's Use Only.* *SCORE:* |

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**2D** At the corner diner, lunches have one set price and dinners have another set price. Five lunches and three dinners cost $70, and three lunches and five dinners cost $74. Determine the cost of one lunch.

**2E** In the following cryptarithm, each letter represents its own digit. Determine the greatest value of the 4-digit number OUCH.

$$\begin{array}{r} \text{STUB} \\ +\quad \text{TOE} \\ \hline \text{OUCH} \end{array}$$

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Registered School: Leota Middle School (WOODINVILLE WA)

Division **E**

# Mathematical Olympiads
## January 14, 2020
*For Elementary & Middle Schools*

Contest **3**

*Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.*

*Name:* ______________________________________________

**3A** Find the sum:

$$\begin{array}{r} 202222 \\ 220222 \\ 222022 \\ 222202 \\ +\ 222220 \\ \hline \end{array}$$

**3B** In the list of numbers 3, 8, 13, 18, …, 98, there are 20 numbers, and each number after the first is five more than the previous number. All of the numbers are added together to give a sum. What is the units digit of the sum?

**3C** In a family, there are four children. Adam’s age is the sum of Beth’s and Carol’s. Four years ago, David’s age was the sum of Beth’s age then and Carol’s age then. Eight years ago, Adam’s age was twice David’s age then. Who is the oldest child in this family?

*Name:* ______________________________________________

| Answer Column |
|---|
| **3A** |
| **3B** |
| **3C** |
| **3D** |
| **3E** |
| *Do Not Write in this Space. For PICO's Use Only.* *SCORE:* |

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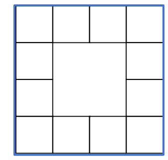
**3D** In the given diagram, a series of identical squares are arranged around a larger square as shown. How many total squares (of any size) are in the diagram using only the existing lines?



**3E** It takes 12 workers exactly 2 hours to build a total of 60 widgets. Working at the same rate, how many widgets could 4 of those workers build in 8 hours?

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Division **E**

# Mathematical Olympiads

**February 11, 2020**

*For Elementary & Middle Schools*

Contest **4**

*Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.*

*Name:* ________________________________________________

**4A** Evaluate: 732 – 935 + 868 – 265.

**4B** A 3-digit counting number has each of the following properties:
* The three digits are in increasing order.
* The sum of the digits is 9.
* The number is a multiple of 15.
What is the number?

**4C** The numbers 10, 11, 12, …, 98, 99 are written as one big 180-digit number (1011121314 … 979899). All the even digits are then removed. How many digits remain?

*Name:* ______________________________

| Answer Column |
| --- |
| **4A** |
| **4B** |
| **4C** |
| **4D** |
| **4E** |
| *Do Not Write in this Space. For PICO's Use Only.* *SCORE:* |

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**4D** What is the least 2-digit counting number that is a multiple of 3, that is one less than a multiple of 4, and that is one more than a multiple of 5?

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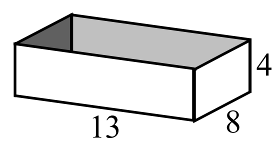
**4E** What is the maximum number of $3 \times 3 \times 2$ bricks that will fit in a box that has dimensions $13 \times 8 \times 4$ without exceeding the dimensions of the box?



Registered School: Leota Middle School (WOODINVILLE WA)

Division **E**

# Mathematical Olympiads
## March 10, 2020
*For Elementary & Middle Schools*

Contest **5**

*Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.*

*Name:* ______________________________________________

**5A** What is the value of $47 \times 12 + 12 \times 53 + 53 \times 8 + 8 \times 47$?

**5B** Don, Jon, and Ron are each thinking of a 2-digit prime number. The numbers are different. Don's number is the least and Ron's number is the greatest. If each of their numbers has digits that total 8, what is Jon's number?

**5C** Sally leaves her house and jogs along a straight road. After one hour she has gone 4 miles. Then, Sally slows to a brisk walk at a constant rate. After the first hour of walking in the same direction, Sally is 6.5 miles from her house. How many miles from her house will she be after two more hours of walking?

*Name:* ______________________________

| Answer Column |
|---|
| **5A** |
| **5B** |
| **5C** |
| **5D** |
| **5E** |
| *Do Not Write in this Space. For PICO's Use Only.* *SCORE:* |

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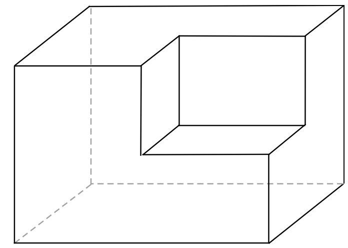
**5D** A rectangular prism (box) has volume 240 cubic cm. A smaller rectangular prism is removed from the larger prism. The lengths of the edges of the smaller prism are one-half the lengths of the corresponding edges of the larger prism. How many cubic centimeters is the volume of the remaining solid?



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**5E** There is an octahedral (8-faces) die whose faces each have one different number from the set {1, 2, 3, 4, 5, 6, 7, 8}. The die is rolled twice, and the number appearing on top after the first roll is multiplied by the number appearing on top after the second roll. What is the probability that this product is divisible by 9?

Division **E** — Mathematical Olympiads November 12, 2019 For Elementary & Middle Schools — Contest **1**

# SOLUTIONS AND ANSWERS

| | |
|---|---|
| **1A** | **29997** |
| **1B** | **Dennis** |
| **1C** | **1647** |
| **1D** | **4** |
| **1E** | **31240** |

**1A** **METHOD 1** *Strategy: Use patterns found in each place value.*
The ones, tens, hundreds, and thousands place digits each consist of 3, 8, 8, and 8 in some order. Using 3 + 8 + 8 + 8 = 27 in each place value, the problem becomes 27 + 270 + 2700 + 27000, which equals **29997**.

**METHOD 2** *Strategy: Use the fact that 3 = 1 + 1 + 1 to simplify the problem.*
In each place value there are three digits that are 8 and one that is a 3. Split each three into three 1's and add one 1 to each 8. The problem is the same as
9990 +9909 + 9099 + 999 = 29997.

*FOLLOW-UP: Find the sum of 6423 + 4236 + 2364 + 3642.* [16665]

**1B** **METHOD 1** *Strategy: Use logical reasoning.*
Since two of the students were 3 away from the actual number, find the two guesses that have a difference of 6. The two guesses belong to Candace and Edward, since 115 – 109 = 6, which means there were 112 marbles. The guess of 113 marbles is closest to 112, therefore **Dennis** won the contest.

**METHOD 2** *Strategy: Make an organized list.*
Add 3 and subtract 3 from each guess to find the two students who had the same guess when they were 3 away from the actual number.

| Student | Guess | +3 | –3 |
|---|---|---|---|
| Ashley | 98 | 101 | 95 |
| Beth | 105 | 108 | 102 |
| Candace | 109 | **112** | 106 |
| Dennis | 113 | 116 | 110 |
| Edward | 115 | 118 | **112** |

The number 112 is 3 more than Candace's guess and 3 less than Edward's guess. There are 112 marbles in the jar, so Dennis is the winner with his guess of 113.

**1C** **METHOD 1** *Strategy: Use number sense and algebra.*
The ones column adds to 27. The tens digits in the three addends is the same as the tens digit in the sum so 3A + 2 equals A + 10 or A + 20. It follows that 2A equals 8 or 18 so A equals 4 or 9. Since A $\neq$ 9, A = 4 and the addends are each 549.
The sum is $3 \times 549$ = **1647**.

**METHOD 2** *Strategy: Use multiplication with trial and error.*
Examine the product of 3 times each addend $3 \times 509 = 1527$, $3 \times 519 = 1557$, $3 \times 529 = 1587$, $3 \times 539 = 1617$, $3 \times 549 = 1647$, etc. Reject the first four possibilities since the tens digit in the product does not equal the tens digit in the addends. The number 1647 satisfies all the conditions.

*FOLLOW-UP: In the following cryptarithm, each letter represents a different digit from 1 through 9. If T is 5, find any possible value for PETS.* [1357 and 1258 are two solutions.]

$$\begin{array}{r} D\,O\,G \\ +\,C\,A\,T \\ \hline P\,E\,T\,S \end{array}$$

**1D** **METHOD 1** *Strategy: Use division and find a pattern.*
Since every six digits of the decimal repeats, divide 2019 by 6 to get 336 with a remainder of 3. Use the remainder to find the 2019$^{th}$ digit of the decimal. The third digit to the right of the decimal is 4, therefore the 2019$^{th}$ digit will be **4**.

**METHOD 2** *Strategy: Use the divisibility test for 6.*
Since 384615 repeats, the digit 5 will repeat in every 6$^{th}$ position. Multiples of 6 are divisible by 2 and 3. Numbers divisible by 3 have the sum of their digits as a multiple of 3. Numbers divisible by 6 have that property but must also be even. The number 2019 is divisible by 3 since $2 + 0 + 1 + 9 = 12$ but 2019 is not even. If we subtract 3 from 2019, we get 2016 which is even and still divisible by 3. Since we want the 2019$^{th}$ digit, it will be 3 digits beyond the digit 5 found in the 2016$^{th}$ position. That digit is 4.

*FOLLOW-UP: The fraction 1/54 is written as a decimal number. What is the 2020$^{th}$ digit to the right of the decimal point?* [5]

**1E** **METHOD 1** *Strategy: Use divisibility rules.*
If a number is divisible by 5, then the units digit must be a 0 or 5. Since 5 is not one of the usable digits, the 5-digit number must end in a 0. If a number is divisible by 11, then the alternating sum and difference of the digits from left to right must also be divisible by 11 (Example: 95381 is divisible by 11 since $9 - 5 + 3 - 8 + 1 = 0$ which is divisible by 11.). Using the digits 1, 2, 3, and 4, the only possible alternating sum that would be divisible by 11 is 0. To get an alternating sum of 0, the digits 1 and 4 should alternate with 2 and 3. This could result in 12430, 21340, and 31240 (42130 cannot be a possible solution because the first digit is 4). To check which of these is divisible by 8, see if the last three digits of the number are divisible by 8. Since 240 is divisible by 8, the 5-digit number is **31240**.

**METHOD 2** *Strategy: Guess and check.*
Make a list of possible 5-digit numbers that fit the parameters of the problem. The units digit must be 0 in order for the number to be divisible by 5 and even. Consider the expansion of the 5-digit number abcde: $abcde = 10000a + 1000b + 100c + 10d + e$. Since the place values 10000 and 1000 are each divisible by 8 the number $10000a + 1000b$ is divisible by 8. Determine which 3-digit numbers using the given digits are divisible by 8: 120, 240, and 320. For each of these there are at most two possible choices for the digits a and b. Test to see which of the numbers is also divisible by 11: 34120, 13240, 31240, and 14320. The only one divisible by 11 is 31240.

*FOLLOW-UP: Find the value of the digit A in the number 3020202A that would make the number divisible by 9 and by 11.* [9]

> ***NOTE:*** *Other FOLLOW-UP problems related to some of the above can be found in our four contest problem books and in Creative Problem Solving in School Mathematics and Teaching Mathematical Problem Solving. Visit www.moems.org for details and to order.*

# SOLUTIONS AND ANSWERS

| | |
|---|---|
| **2A** | **44** |
| **2B** | **8** |
| **2C** | **20190** |
| **2D** | **$8** |
| **2E** | **9543** |

**2A** **METHOD 1** *Strategy: Select optimal groupings.*
(98 – 87) + (76 – 65) + (54 – 43) + (32 – 21) = 11 + 11 + 11 + 11 = 4 × 11 = **44**.

**METHOD 2** *Strategy: Group the "plusses" and the "minuses"; then, subtract.*
Plusses: 98 + 76 + 54 + 32 = (90 + 70 + 50 + 30) + (8 + 6 + 4 + 2) = 240 + 20 = 260
Minuses: 87 + 65 + 43 + 21 = (80 + 60 + 40 + 20) + (7 + 5 + 3 + 1) = 216
Subtract: 260 – 216 = 44

**METHOD 3** *Strategy: Work the tens and ones columns separately.*
10 × ((9 – 8) + (7 – 6) + (5 – 4) + (3 – 2)) = 40
(8 – 7) + (6 – 5) + (4 – 3) + (2 – 1) = 4
Therefore 98 – 87 + 76 – 65 + 54 – 43 + 32 – 21 = 44.

*FOLLOW-UP: Evaluate: 91 – 82 + 73 – 64 + 55 – 46 + 37 – 28.* [36]

**2B** **METHOD 1** *Strategy: Recall rules for adding odds and evens.*
For a sum to be even, there needs to be an even number of odd numbers (and it does not matter how many even numbers there are). The only 1 × 1 squares are 2, 4, 6, and 8. Then, all of the 2 × 2 squares have two odds apiece, on a diagonal. Note: the 3 × 3 square has five odd numbers and would not have an even number sum. So, there is a total of 4 + 4 + 0 = **8** squares.

**METHOD 2** *Strategy: Carefully remove rows and columns from the whole figure.*
Draw the squares that meet the criteria.

| | |
|---|---|
| 1 | 2 |
| 4 | 5 |

| | |
|---|---|
| 2 | 3 |
| 5 | 6 |

| | |
|---|---|
| 5 | 6 |
| 8 | 9 |

| | |
|---|---|
| 4 | 5 |
| 7 | 8 |

| |
|---|
| 2 |

| |
|---|
| 4 |

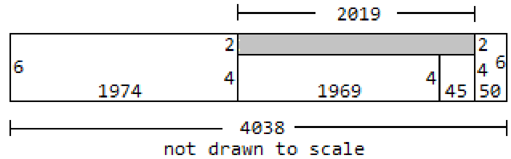
| |
|---|
| 6 |

| |
|---|
| 8 |

*FOLLOW-UPS: (1) For the same figure, how many rectangles (of any size) can be drawn along existing lines such that the number or the number sum inside the rectangle is even?* [12] *or odd?* [24]
*(2) In the figure, how many squares (of any size) can be drawn along existing lines, such that the number or the number sum inside the square is divisible by 3?* [6]

**2C** **METHOD 1** *Strategy: Set up "like multiples" (distributive factors) in the process.*
(1974 × 6) + (45 × 6) + (1969 × 4) + (50 × 4) = (1974 + 45) × 6 + (1969 + 50) × 4 = 2019 × 6 + 2019 × 4 = 2019 × (6 + 4) = 2019 × 10 = **20190**.

**METHOD 2** *Strategy: Visualize products as areas and rearrange them into a convenient shape.*
Lining up the products (as areas), we *almost* get a 4038 × 6 rectangle.
The area of the shaded rectangle (2019 × 2 = 4038) needs to be removed.
In total, (4038 × 6) – (4038 × 1) = 4038 × (6 – 1) = 4038 × 5 = (2 × 2019) × (½ × 10) = (2 × ½) × (2019 × 10) = 1 × 20190 = 20190.




*FOLLOW-UPS: (1) Evaluate 576 × 17 + 582 × 15 + 1424 × 17 + 1418 × 15.* [64,000]
*(2) Evaluate (1976/6) + (1974/3) + (1969/2) + (43/6) + (45/3) + (50/2).* [2019]

**2D** **METHOD 1** *Strategy: Find the difference in cost between a lunch and a dinner.*
If turning two lunches into dinners raises the cost \$4 then a dinner must cost \$2 more than a lunch.
Then turning three dinners into lunches should reduce the cost by \$6.
It follows that eight lunches should cost \$64.
Therefore, one lunch should cost \$**8**.

**METHOD 2** *Strategy: Combine the information to set up a symmetry.*
In total, there are (5 + 3) = 8 lunches, (3 + 5) = 8 dinners, and together the cost is (\$70 + \$74) = \$144.
Divide this by 8 to get that 1 lunch + 1 dinner = \$18. So, 3 lunches + 3 dinners = (3 × \$18) = \$54.
Compare this with the cost of 5 lunches + 3 dinners = \$70 to see that the extra (5 – 3) = 2 lunches cost (\$70 – \$54) = \$16, or that one lunch costs \$8.

*FOLLOW-UP: At the down-the-block diner, lunches have one set price and dinners have another set price. If four lunches and three dinners cost \$58, and five lunches and four dinners cost \$75, determine the cost of one lunch.* [\$7]

**2E** **METHOD 1** *Strategy: Identify necessary relationships in the addition (such as regrouping).*
To maximize OUCH, let O = 9, so S = 8. U cannot be 7 or 6 because it would make T = 8 = S. Let U = 5, so T = 7. In the tens place, U + O = C becomes 5 + 9 = 14, so C = 4. Of the remaining digits: 0, 1, 2, 3, and 6, only H = 3 (from 1 + 2) works. Thus, OUCH = **9543**.

**METHOD 2** *Strategy: Try highest values first. Test cases.*
First try: OUCH = 9876 which results in S = U = 8. (In fact, if O = 9, then S = 8.) Second try: OUCH = 9765 with S = 8. But the only way to get U = 7 is if T = 8 (which is already S) or T = 3 (which fails to cause regrouping in the thousands place). Third try: OUCH = 9675 with S = 8. Similarly, there is no good value for T. Fourth try: OUCH = 9576 with S = 8. Here, T = 7 would work if C wasn't 7. So, adjust to try OUCH = 9564 with S = 8 and T = 7. Here, 875B + 79E = 9564. That doesn't work. Adjust again to try OUCH = 9546 with S = 8 and T = 7. Here, 875B + 79E = 9546. There are no possible pairs of B and E that work. Adjust again to try OUCH = 9543 with S = 8 and T = 7. Here, 875B + 79E = 9543, which works because B and E can be 1 and 2, in some order.

*FOLLOW-UP: Knowing that 0 can never be a starting digit of a number, find the least value of OUCH in the given cryptarithm.* [2034]

***NOTE:*** *Other FOLLOW-UP problems related to some of the above can be found in our four contest problem books and in Creative Problem Solving in School Mathematics and Teaching Mathematical Problem Solving. Visit www.moems.org for details and to order.*

| Division | Mathematical Olympiads | Contest |
|---|---|---|
| E | January 14, 2020 For Elementary & Middle Schools | 3 |

# SOLUTIONS AND ANSWERS

| | |
|---|---|
| 3A | 1088888 |
| 3B | 0 |
| 3C | Adam |
| 3D | 18 |
| 3E | 80 |

**3A** **METHOD 1** *Strategy: Find a pattern and use place value to find the sum.*
There are four 2's in the ones, tens, hundreds, thousands, and ten-thousands columns. The hundred-thousands column contains five 2's. The sum of each of the ones, tens, hundreds, thousands, and ten-thousands column is 8. The hundred-thousands column sums to 10. The resulting sum is **1088888**.

**METHOD 2** *Strategy: Apply the standard algorithm for addition.*

$$\begin{array}{r} 202222 \\ 220222 \\ 222022 \\ 222202 \\ +\ 222220 \\ \hline 1088888 \end{array}$$

*FOLLOW-UP: Find the sum 505555 + 550555 + 555055 + 555505 + 555550.* [2722220]

**3B** **METHOD 1** *Strategy: Reason using number sense.*
There are 20 numbers in the list and each subsequent number increases by 5, resulting in a pattern of 3, 8, 3, 8, 3, 8 … in the units place. Add 3 + 8 = 11. Since there are 10 equal groups of 11, the units digit will be the units digit of the product 10 × 11 = 110, which equals **0**.

**METHOD 2** *Strategy: Find the sum of the numbers and identify the units digit.*
A quick way to add all the numbers in the list is to write the numbers twice and add:

$$\begin{array}{cccccccc} 3\ + & 8\ + & 13\ + & 23\ + \ldots + & 83\ + & 88\ + & 93\ + & 98 \\ 98\ + & 93\ + & 88\ + & 83\ + \ldots + & 23\ + & 13\ + & 8\ + & 3 \\ \hline 101\ + & 101\ + & 101\ + & 101 + \ldots + & 101\ + & 101\ + & 101\ + & 101 \end{array}$$

The total will equal 20 × 101 = 2020. This is twice the total we need or 1010. Therefore, the units digit is 0.

*FOLLOW-UPS: (1) Find the units digit of the sum of all the positive multiples of 4 that are less than 100.* [0]
*(2) Find the tens digit for the sum in part (1).* [0]

**3C** **METHOD 1** *Strategy: Develop an organized list.*
Adam's age = Beth's + Carol's ages
David's age = Beth's + Carol's ages (4 years ago)
Adam's age = 2 × David's age (8 years ago)
Therefore, Beth and Carol are younger than both Adam and David. Since Adam's age was twice David's age 8 years ago, **Adam** must be the oldest child.

**METHOD 2** *Strategy: Use the process of elimination.*
Could David be the oldest child? No, because Adam was twice David's age 4 years ago.
Could Carol be the oldest child? No, since Adam's age is the sum of Carol's and Beth's ages.
Could Beth be the oldest child? No, due to the reasoning above.
Thus, Adam must be the oldest child.

*FOLLOW-UP: If Beth and Carol are twins, what are their current ages?* [8]

**3D** **METHOD 1** *Strategy: Count the number of each of the different sized squares.*
The number of $1 \times 1$ squares is 12.
The number of $2 \times 2$ squares is 1 (the inner square).
The number of $3 \times 3$ squares is 4.
The number of $4 \times 4$ squares is 1 (the outermost square).
The total number of squares is $12 + 1 + 4 + 1 =$ **18**.

**METHOD 2** *Strategy: Consider 3 different location types.*
Count the number of squares that contain the upper left corner $1 \times 1$ square: there is the square itself, a $3 \times 3$, and a $4 \times 4$ square, for a total of 3 squares. Each corner square has the same property, but we do not wish to count the $4 \times 4$ square more than once. Therefore, there are $3 + 2 + 2 + 2 = 9$ squares that contain corner squares.
Count the number of squares that contain an edge square but no corner square: there are the 8 squares each measuring $1 \times 1$.
Finally, the last square to count is one that does not contain either an edge or a corner square: there is only the one $2 \times 2$ square in the middle. Thus, we have a total of $9 + 8 + 1 = 18$ squares.

*FOLLOW-UP: Find the total number of rectangles in the diagram.* [52]

**3E** **METHOD 1** *Strategy: Apply proportional reasoning.*
Find the rate for each worker, 60 widgets/12 workers = 5 widgets/worker in 2 hours. This means that each worker builds 5 widgets in 2 hours or 20 widgets in 8 hours. Therefore, 4 workers can build $4 \times 20 =$ **80** widgets in 8 hours.

**METHOD 2** *Strategy: Make a table.*
Make problem simpler by dividing by 3 to create a chart for 4 workers only.

| Number of Workers | Cumulative Hours | Number of Widgets completed every two hours |
|---|---|---|
| 4 | 2 | 20 |
| 4 | 4 | 40 |
| 4 | 6 | 60 |
| 4 | 8 | 80 |

*FOLLOW-UP: Find the number of minutes it would take 1 worker to build 1 widget.* [24 minutes]

***NOTE:*** *Other FOLLOW-UP problems related to some of the above can be found in our four contest problem books and in Creative Problem Solving in School Mathematics and Teaching Mathematical Problem Solving. Visit www.moems.org for details and to order.*

Division **E**

# Mathematical Olympiads
**February 11, 2020**
*For Elementary & Middle Schools*

Contest **4**

## SOLUTIONS AND ANSWERS

**4A** **METHOD 1** *<u>Strategy</u>: Combine the two numbers being added then combine the two numbers being subtracted.*
732 + 868 = 1600
935 + 265 = 1200
Subtract the second result from the first: 1600 – 1200 = **400**.

**METHOD 2** *<u>Strategy</u>: Reorder terms and compute from left to right.*
(732 + 868) – 935 – 265 = (1600 – 935) – 265 = 665 – 265 = 400.

*FOLLOW-UP: Mr. Bear had $157 in the bank. He borrowed enough money to buy something for $249. That week he earned $529. He paid off his debt and put the rest in the bank. How much does he have in the bank?* [$437]

**4B** **METHOD 1** *<u>Strategy</u>: Apply logical reasoning.*
The number is a multiple of 15 so the units digit must be 0 or 5. Since the digits appear in increasing order, the units digit cannot be 0 so it must be five. Since the three digits sum to 9, and the units digit is already 5, the hundreds and tens digits must sum to 4. The digits must be different in order to be in increasing order so the two digits must be 1 and 3. Therefore, the solution must be **135**.

**METHOD 2** *<u>Strategy</u>: Make an organized list.*
The only 3-digit numbers whose digits add up to 9 and that are in increasing order are: 126, 234, and 135. The only one that is a multiple of 15 is 135.

*FOLLOW-UP: What is the <u>greatest</u> 4-digit number that satisfies the following properties: (a) Its four digits are in decreasing order. (b) The sum of the digits is 18. And (c) The number is a multiple of 15.* [9810]

**4C** *<u>Strategy</u>: Solve a simpler but similar problem to find a pattern.*
There are 9 sets of two-digit numbers from 10 to 99. This makes 180 digits.
The first set of 2-digit numbers beginning with an odd number ("odd beginners") is **1**~~0~~, **11**, **1**~~2~~, **13**, **14**, **15**, **1**~~6~~, **17**, **1**~~8~~, **19**. Count odd digits only. There are 15. There are 5 sets of two-digit "odd beginners," making 75 odd digits.
The first set of 2-digit numbers beginning with an even number ("even beginners") is ~~20~~, ~~2~~**1**, ~~22~~, ~~2~~**3**, ~~24~~, ~~2~~**5**, ~~2~~6, ~~2~~**7**, ~~28~~, ~~2~~**9**. Count odd digits only. There are 5.
There are 4 sets of two-digit "even beginners," making 20 odd digits. Thus, there are 75 + 20 = **95** odd digits of the 180 digits.

*FOLLOW-UP: The numbers 100, 101, 102, …, 198, 199 are written as one big number. All the odd digits are then removed. How many digits remain?* [100]

| | |
|---|---|
| **4A** | **400** |
| **4B** | **135** |
| **4C** | **95** |
| **4D** | **51** |
| **4E** | **20** |

**4D** **METHOD 1** _Strategy_: *Use divisibility rules.*

There are fewer multiples of 5 than of 2 or 3. So, begin with the clue related to the multiples of 5. Multiples of 5 have 0 or 5 in the units place. Numbers that are one more than a multiple of 5 need to end in a 1 or a 6. Multiples of 4 are even numbers. One less than a multiple of 4 would be an odd number. Thus, the number must have 1 in the units place. Multiples of 3 have digits that add to 3 or a multiple of 3. Since we want the least number that has these attributes, list the two-digit numbers from least to greatest that have a 1 in the units place whose digits add to a multiple of 3: 21, 51, 81. Of these, all are multiples of 3 and all are one more than a multiple of 5. The only one that is one less than a multiple of 4 is **51**.

**METHOD 2** _Strategy_: *Make organized lists.*

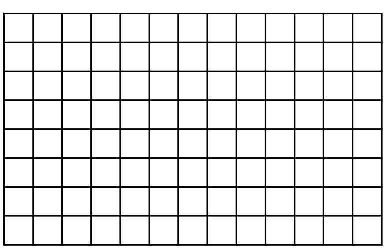
The shortest list will be the 2-digit numbers that are 1 more than a multiple of 5:
11, 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81, 86, 91, 96.
Cross out those that are not 1 less than a multiple of 4:
11, ~~16~~, ~~21~~, ~~26~~, 31, ~~36~~, ~~41~~, ~~46~~, 51, ~~56~~, ~~61~~, ~~66~~, 71, ~~76~~, ~~81~~, ~~86~~, 91, ~~96~~ → 11, 31, 51, 71, 91.
Cross out those that are not a multiple of 3:
~~11~~, ~~31~~, 51, ~~71~~, ~~91~~ → The only number not eliminated is 51.

*FOLLOW-UP: How many 3-digit numbers are there that are one more than a multiple of 9 and 1 less than a multiple of 10?* [10]

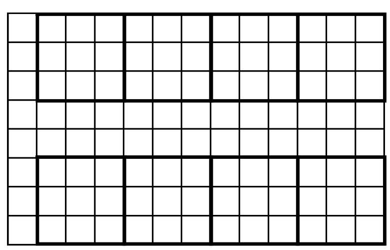
**4E** **METHOD 1** _Strategy_: *Apply spatial reasoning.*

A bottom layer of $3 \times 3 \times 2$ (height) gives 4 bricks along the long side and 2 bricks along the width. This leaves a unit of 1 on the length and 2 on the width. We now have 8 bricks on the bottom layer. We have room for another layer in height, for 16 bricks. Finally, along the width we have a $2 \times 3 \times 3$. This will fit another 4 bricks. In total **20** bricks can fit in the box.

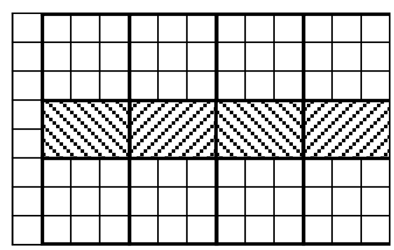
**METHOD 2** _Strategy_: *Visualize the possible arrangements of the bricks.*



Visualize the area of the bottom of the box.



Place as many bricks across the bottom with the $3 \times 3$ side face down. (8 bricks)



Fill in the empty space with bricks using the $2 \times 3$ side face down. (4 bricks)

Since we can fit 2 layers of bricks with the $3 \times 3$ side face down and a single layer of bricks with the $2 \times 3$ side face down, we can fit a total of $2(8) + 4 = 20$ bricks in the $13 \times 8 \times 4$ box.

*FOLLOW-UP: Using the 24 bricks with dimensions $3 \times 3 \times 2$, compute the difference between the greatest area that can be covered by the bricks and the least area that can be covered by the bricks.* [72 sq. units]

***NOTE:*** *Other FOLLOW-UP problems related to some of the above can be found in our four contest problem books and in Creative Problem Solving in School Mathematics and Teaching Mathematical Problem Solving. Visit* _www.moems.org_ *for details and to order.*

# SOLUTIONS AND ANSWERS

| | |
|---|---|
| 5A | 2000 |
| 5B | 53 |
| 5C | 11.5 |
| 5D | 210 |
| 5E | 1/16 |

**5A** **METHOD 1** *Strategy: Regroup and simplify.*

Regroup: $12 \times (47 + 53) + 8 \times (47 + 53) = (12 \times 100) + (8 \times 100)$

$= 1200 + 800 =$ **2000**

**METHOD 2** *Strategy: Follow the order of operations.*

$47 \times 12 + 12 \times 53 + 53 \times 8 + 8 \times 47 = 564 + 636 + 424 + 376 = 2000$

*FOLLOW-UPS: (1) What is the value of $47 \times 12 + 12 \times 53 - 53 \times 8 - 8 \times 47$?* [400]

*(2) What is the value of $33 \times 12 + 12 \times 66 - 66 \times 8 - 8 \times 33$?* [396]

**5B** **METHOD 1** *Strategy: List all 2-digit whole numbers whose digit sum is 8. Make an ordered sub list of the primes.*

The 2-digit whole numbers with digit sum equal to 8: 17, 26, 35, 44, 53, 62, 71, 80. The primes from this list are: 17, 53, 71.

From sentence #2: Don's # < Jon's # < Ron's #, therefore Jon's number is **53**.

**METHOD 2** *Strategy: Use logic and number sense.*

Since all the numbers are prime, eliminate any even two-digit numbers. List the odd two-digit numbers that add up to 8: 17, 35, 53, 71. Thirty-five is not prime (multiple of 5) so the numbers are 17, 53 and 71. Ron's is the greatest (71), Don's number is the least (17) and Jon's number must be 53.

*FOLLOW-UPS: Given the original problem, if the digit sum is 10, what is Jon's number?* [37]

**5C** **METHOD 1** *Strategy: Use the formula: distance = rate × time.*

Sally jogs 4 miles in 1 hour. Walking, Sally covers an additional 2.5 miles in the next hour. Therefore, Sally's walking rate is 2.5 mph. After 2 more hours of walking, Sally covers an additional 2 hours × 2.5 mph = 5 miles. Altogether, Sally has covered 4 + 2.5 + 5 = **11.5 miles** in the 4 hours.

**METHOD 2** *Strategy: Make a table.*

| | | | | |
|---|---|---|---|---|
| Distance (miles) | 4 | 2.5 | 2.5 | 2.5 |
| Time (hours) | 1 | 1 | 1 | 1 |

Sally covered 4 + 2.5 + 2.5 + 2.5 = 11.5 miles in the 4 hours.

*FOLLOW-UPS: (1) What was Sally's average rate of speed during her entire journey?* [2.875 mph]
*(2) Suppose Sally took a 30-minute break after jogging the initial 4 miles. Then, Sally walks briskly at a constant rate. After the first hour of walking in the same direction, Sally is 6.5 miles from her house. How far from home would she be after 4 hours?* [10.25 miles]

**5D** **METHOD 1** <u>Strategy</u>: *The volume of a rectangular prism: volume $=$ length $\times$ width $\times$ height.*
Let L, W, and H be the number of centimeters in the length, width, and height, respectively, of the original large rectangular prism. The dimensions of the cutout are $\frac{1}{2}L$, $\frac{1}{2}W$, and $\frac{1}{2}H$, which yields the volume $\frac{1}{2}L \times \frac{1}{2}W \times \frac{1}{2}H = \frac{1}{8}LWH = \frac{1}{8} \times 240 = 30$ cubic cm. The remaining volume is $240 - 30 =$ **210** cm$^3$.

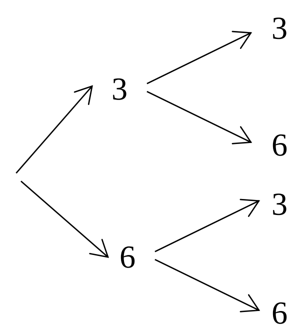
**METHOD 2** <u>Strategy</u>: *Assume even-numbered dimensions for length, width, and height.*
Even-numbered dimensions for a 240 cubic cm rectangular prism could be length = 12, width = 10, and height = 2. (Volume = length × width × height.) Halving each of those dimensions, we get a volume of 6 × 5 × 1 = 30 cubic cm. Subtracting the removed volume, 240 – 30 = 210 cm$^3$. (Similarly, we could have chosen 4, 6, and 10; or 2, 4, and 30; or 2, 2, and 60; or 2, 6, and 20 for the original dimensions).

**METHOD 3** <u>Strategy</u>: *Use a visual examination of the diagram.*
Observe that the smaller prism is 1/8 of the entire prism. Therefore, the remaining volume will be 7/8 of the entire volume and (7/8) × 240 = 210 cm$^3$.

*FOLLOW-UPS: Use the original problem and let each of the dimensions of the cutout be a positive integer:*
*(1) What is the least possible surface area of the cutout rectangular prism?* [62 square cm]
*(2) What is the greatest possible surface area of the cutout rectangular prism?* [122 square cm]

**5E** **METHOD 1** <u>Strategy</u>: *Examine the possible outcomes.*
There are 8 possible outcomes on the first roll, and 8 possible outcomes on the second, so there are 64 (8 × 8) possible outcomes with two rolls of the die (1-1, 1-2, 1-3, …, 2-1, 2-2, 2-3, …, 2-7, 2-8, …, etc.) The products that would be divisible by 9 are 9 (3 & 3), 18 (3 & 6 or 6 & 3) and 36 (6 & 6). The probability of getting the rolls that produce a number divisible by 9 is 4/64 or $\mathbf{\frac{1}{16}}$.

**METHOD 2** <u>Strategy</u>: *Use your knowledge of probability to make a tree diagram.*
For a number to be a multiple of 9, it must have 2 or more factors of 3. The only numbers on the faces of the octahedral die which have factors of 3 are: 3 and 6. Therefore, each roll of the die must be either 3 or 6. Construct a tree using this idea. At each branch, the probability of rolling the number shown is $\frac{1}{8}$. The four ends of the paths resulting from rolling 3, 3 or 3, 6 or 6, 3 or 6, 6 each have probability $\frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$. Adding these probabilities yields the total probability:
$\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} = \frac{4}{64} = \frac{1}{16}$.




*FOLLOW-UPS: (1) Beginning with the same octahedral die, what is the probability of rolling the die twice and having the product of the two values be a perfect square?* [3/16]
*(2) Using the original problem, what is the probability that the product is* ***not*** *divisible by 9?* [15/16]

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